MATH 401-Section 0201

APPLICATIONS OF LINEAR ALGEBRA

CHAPTER 1 - Part A.
LINEAR ALGEBRAIC SYSTEMS

FALL - 2014

Solution of Linear Systems

Investment Portfolio

You have a portfolio totaling $\$200.000$ and want to invest in **municipal bonds**, **blue-chip stocks**, and **speculative stocks**. The municipal bonds pay 6% annually. Over a 5-years period you expect blue-chip stocks to return 10% annually and speculative stocks to return 15% annually. You want a combined annual return of 8%, and you also want to have only one-fourth of the portfolio invested in stocks. How much should be allocated to each type of investment?

Solution

Let M represent municipal bonds, B represent blue-chip stocks, and G represent speculative stocks.

$$\begin{cases} M + B + G = 200.000 & \textbf{Eq 1}\text{: total investment is } 200.000 \\ 0.06M + 0.10B + 0.15G = 16.000 & \textbf{Eq 2}\text{: combined annual return } 8\% \text{ of } 200.000 \\ B + G = 50.000 & \textbf{Eq 3}\text{: } \frac{1}{4} \text{ of investment is allocated to stocks} \end{cases}$$

- **3 equations and 3 unknowns**

Solution of Linear Systems

Definition: System of Linear Equations.

A "system" of equations is a set or collection of equations that you deal with all together at once. The following is a **lineal system of m equations and n unknowns**:

$$\left.\begin{array}{llllll} E_1: & a_{11}x_1+ & a_{12}x_2+ & \cdots & +a_{1n}x_n = & b_1 \\ E_2: & a_{21}x_1+ & a_{22}x_2+ & \cdots & +a_{2n}x_n = & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \\ E_m: & a_{m1}x_1+ & a_{m2}x_2+ & \cdots & +a_{mn}x_n = & b_m \end{array}\right\} \quad \text{(S)}$$

where, for $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, n\}$, the constants a_{ij} are the **coefficients**, b_i the **constant terms** and $x_1, \cdots, x_n$ are the **unknown of the system**. First we consider that there are the same number of equations and unknowns ($m = n$).

Solution of Linear Systems

Gaussian Elimination

Gaussian Elimination is a simple, systematic algorithm to solve systems of linear equation.

Goal

The goal of this method is to weed out selective entries a_{ij} (or coefficients) of the system by performing linear combination of equations.

Example

$$\begin{array}{rl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = & b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = & b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = & b_3 \end{array} \Longleftrightarrow \begin{array}{rl} \widetilde{a}_{11}x_1 + \widetilde{a}_{12}x_2 + \widetilde{a}_{13}x_3 = & \widetilde{b}_1 \\ \widetilde{a}_{22}x_2 + \widetilde{a}_{23}x_3 = & \widetilde{b}_2 \\ \widetilde{a}_{33}x_3 = & \widetilde{b}_3 \end{array}$$

$(Triangular form)$

Solution of Linear Systems

Gaussian Elimination

We use three operations to simplify the linear system (S):

1. Equation E_i can be multiplied by any nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \to (E_i)$
2. Equation E_j can be multiplied by any nonzero constant λ and added to the equation E_i with the resulting equation used in place of E_i. This operation is denoted $(E_i + \lambda E_j) \to (E_i)$
3. Equations E_i and E_j can be transposed in order. This operation is denoted $(E_i) \leftrightarrow (E_j)$

By a sequence of these operations, a linear system can be transformed to a more easily solved linear system that has the same solution.

Solution of Linear Systems

Example To illustrate, consider a system of three linear equations

$$\left.\begin{aligned} x+2y+z &= 2 \\ 2x+6y+z &= 7 \\ x+y+4z &= 3 \end{aligned}\right|$$

Solution

$$\left.\begin{aligned} x+2y+z &= 2 \\ 2x+6y+z &= 7 \\ x+y+4z &= 3 \end{aligned}\right| \sim \left.\begin{aligned} x+2y+z &= 2 \\ 2y-z &= 3 \\ z &= 1 \end{aligned}\right|$$

We solved the triangular system by the method of Back Substitution:

$$x=-3, \quad y=2, \quad z=1.$$

Matrices

Definición: Matrix

A matrix is a rectangular array of numbers. We use the notation

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

or $A = (a_{ij})$ for a general matrix of size $m \times n$, where m denotes the number of rows in A and n denotes the number of columns.

Example

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1.4 & -22 & 0.5 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2.5 & 1 \\ 2 & -8 \end{pmatrix}.$$

Matrices

Definition

- The set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$.
- **Matrix Addition**.
  Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$, then the sum $A + B$ is calculated entrywise:

$$A + B = (c_{ij}) \quad \text{with} \quad c_{ij} = a_{ij} + b_{ij}, \forall i = 1, ..., m, \forall j = 1, ..., n.$$

- **Multiplication by Scalars**
  Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $\alpha \in \mathbb{R}$ the the scalar multiplication αA is given by multiplying every entry of A by c:

$$\alpha A = \alpha(a_{ij}) = (\alpha a_{ij}).$$

Definition

- **Matrix Product**

  Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, $B = (b_{ij}) \in \mathbb{R}^{n\times p}$. The matrix product $C = A \cdot B$ belongs to $\mathbb{R}^{m\times p}$, con

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Even if both products are defined, they need not be equal, i.e., generally one has $AB \neq BA$.

Example

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Example

- Matrix-Matrix product

If $A = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 7 & 8 \\ 4 & 6 & 0 \\ 5 & 7 & 3 \end{pmatrix}$.

Then, only AB is possible and we obtain:

$$AB = \begin{pmatrix} 27 & 55 & 31 \\ 17 & 49 & 36 \end{pmatrix}$$

Matrices

Definition

- The null matrix is a $m \times n$ matrix consisting of all zero entries. We denote this matrix by θ.
- We denote the identity matrix of order n as I_n or simply by I if the size can be trivially determined by the context. I_n is a square matrix with ones on the main diagonal and zeros elsewhere, moreover

$$A \cdot I = I \cdot A = A, \quad \forall A \in \mathbb{R}^{n \times n}.$$

Example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices

Definition: Matrix Inverse.

An $n \times n$ square matrix $A \in \mathbb{R}^{n\times n}$ is called invertible (also nonsingular or nondegenerate) if there exits $B \in \mathbb{R}^{n\times n}$ such that $A \cdot B = I$ and $B \cdot A = I$,

- Matrix B is called **inverse of A**, denoted by A^{-1}.
- If A is invertible, then the inverse A^{-1} is uniquely determined.
- If A and B are invertibles, then AB is invertible and

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}.$$

Solution of Linear Systems

Back to the system

$$\left.\begin{array}{rr} x + 2y + z = & 2 \\ 2x + 6y + z = & 7 \\ x + y + 4z = & 3 \end{array}\right|$$

We can write the above system as $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

Solution of Linear Systems

Remark

System (S) can be written as follow:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A=\text{coefficients matrix}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X=\text{unknowns}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{B=\text{independent terms}},$$

and we obtain the matrix form:

$$AX = B\,.$$

- If $b_1 = b_2 = \cdots = b_m = 0$, the above matrix equation is called **homogeneous** , and **non-homogeneous** or **inhomogeneous** otherwise.

Solution of Linear Systems

Gaussian Elimination. Regular Case

We begin by replacing the system (S) by its matrix constituents $AX = B$. For the purpose of performing the same elementary row operations in A and B we introduce the Augmented matrix.

Definition

Given the system $AX = B$, the **augmented matrix** is given by $M = (A|B) \in \mathbb{R}^{m\times(n+1)}$

$$M = (A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & .... & a_{1n} & b_1 \\ a_{21} & a_{22} & .... & a_{2n} & b_2 \\ \vdots & \vdots & .... & \vdots & \vdots \\ a_{m1} & a_{m2} & .... & a_{mn} & b_m \end{array}\right)$$

Solution of Linear Systems

Example

- System of linear equation: $$\begin{array}{rr} x+2y+z= & 2 \\ 2x+6y+z= & 7 \\ x+y+4z= & 3 \end{array}$$

- Matrix form $A\mathbf{x}=\mathbf{b}$ with

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad \mathbf{x}=\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{b}=\begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

- Augmented matrix $M=(A|\mathbf{x})=\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right)$

Solution of Linear Systems

Example (Cont.)

By applying elementary row operations in M we obtain:

$$M = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 6 & 1 & 7 \\ 1 & 1 & 4 & 3 \end{array}\right) \sim \quad N = (U|\mathbf{c}) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & \frac{5}{2} & \frac{5}{2} \end{array}\right)$$

Then (1) is equivalent to $U\mathbf{x} = \mathbf{c}$, where the coefficient matrix U is upper triangular, namely, $u_{ij} = 0$ whenever $i > j$.

Definition

A square matrix A will be called regular if the algorithm successfully reduces it to upper triangular U with all non-zero pivots on the diagonal.

Solution of Linear Systems

Elementary Matrices

A key observation is that elementary row operations can be realized by matrix multiplication.

Example.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow{(R_2 - 2R_1) \to R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

Solution of Linear Systems

Example.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow[(R_3 - R_1)\to R_3]{(R_2 - 2R_1)\to R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \xrightarrow{(R_3 + \frac{1}{2}R_2)\to R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}$$

If we set $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$

we obtain that when

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \quad \text{then} \quad E_3E_2E_1A = U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{pmatrix}.$$

Solution of Linear Systems

Inverse elementary matrix.

To undo the operation of adding c times row j to row i, we must perform the inverse row operation that subtract c times row j from row i.

Example.

If

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

then

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{L_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

Solution of Linear Systems

Inverse elementary matrix.

To undo the operation of adding c times row j to row i, we must perform the inverse row operation that subtract c times row j from row i.

Example.

The matrices L_1, L_2 and L_3 defined by

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

are the inverses of E_1, E_2 and E_3, respectively, namely

$$L_1E_1 = L_2E_2 = L_3E_3 = I.$$

Example

Moreover

$$L = L_1 L_2 L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{pmatrix}.$$

Here L is a lower triangular matrix with all the entries on the diagonal are equal to 1 (unit lower triangular matrix).

Lemma.

If L and $\widehat{L}$ are lower triangular matrices of the same size, so is their product $L\widehat{L}$. Similarly, if U and $\widehat{U}$ are upper triangular matrices of the same size, so is their product $U\widehat{U}$.

Solution of Linear Systems

The LU Factorization

From the above example we notice that

$$
\begin{aligned}
LU =& (L_1L_2L_3)(E_3E_2E_1A) = L_1L_2(L_3E_3)E_2E_1A = L_1L_2IE_2E_1A \\
=& L_1(L_2E_2)E_1A = L_1IE_1A = (L_1E_1)A = IA = A.
\end{aligned}
$$

Conclusions

- $A = LU$

$$
\left.\begin{array}{ll} L: & \text{unit lower triangular} \\ U: & \text{upper triangular} \end{array}\right\} LU \text{ Decomposition (of factorization) of } A.
$$

- This procedure is dimension-independent, namely it works for matrices $A \in \mathbb{R}^{n\times n}$ for as long as A has $n-1$ nonvanishing pivots.

Solution of Linear Systems

The LU Factorization

- If $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, then

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \iff \boldsymbol{L}(\boldsymbol{U}\boldsymbol{x}) = \boldsymbol{b} \iff \begin{cases} \boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}, \\ \boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}. \end{cases}$$

  Therefore, to solve the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is equivalent to:

  1. Solve $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$ and, then,
  2. Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$.

Both are triangular systems, lower and upper, respectively.

Solving triangular systems

- Given

$$\boldsymbol{L} = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \quad \text{and} \quad \boldsymbol{U} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{pmatrix}$$

  $\boldsymbol{L}$ is lower triangular and $\boldsymbol{U}$ is upper triangular.

- Triangular system solving is easy because the unknowns can be resolved without any further manipulation of the matrix of coefficients.

Example

Consider the following 3-by-3 lower triangular case:

$$\boldsymbol{L} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

The unknowns can be determined as follows:

$$\begin{aligned} x_1 &= b_1/l_{11} \\ x_2 &= (b_2 - l_{21}x_1)/l_{22} \\ x_3 &= (b_3 - l_{31}x_1 - l_{32}x_2)/l_{33} \end{aligned}$$

This is the 3-by-3 version of an algorithm known as forward substitution. Notice that the process requires l_{11}, l_{22}, l_{33} to be nonzero.

Solution of Linear Systems

Forward Substitution

- Consider the system $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{b}$ with lower triangular matrix $\boldsymbol{L}$. We proceed by simple forward substitution of variables:

$$\begin{cases} l_{11}x_1 & = & b_1 & \Rightarrow & x_1 = b_1/l_{11} \\ l_{21}x_1 + l_{22}x_2 & = & b_2 & \Rightarrow & x_2 = \left(b_2 - l_{21}x_1\right)/l_{22} \\ \vdots & & \vdots & & \\ l_{n1}x_1 + \cdots + l_{nn}x_n & = & b_n & \Rightarrow & x_n = \left(b_n - l_{n1}x_1 - \cdots - l_{nn-1}x_{n-1}\right)/l_{nn} \end{cases}$$

- Algorithm:

For $i = 1, \ldots, n$

$$x_i = \frac{1}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}x_j\right)$$

Backward Substitution

On the other hand, consider the system $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{b}$ with upper triangular con matrix $\boldsymbol{U}$. Then we obtain the following algorithm:

For $i = n, n-1, \ldots, 1$

$$x_i = \frac{1}{u_{ii}} \left(b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right)$$